Efficient representation of in-plane rotation within a PCA framework

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Abstract

In this paper, we derive an analytic representation of the eigenvectors and coefficients for in-plane rotated images. This, on the one hand, allows an efficient PCA-basis calculation in the learning stage and on the other hand a direct computation of the rotation angle in the recognition stage. In the experimental section, we demonstrate that the new method is feasible and the recognition and out-of-plane results are comparable to the standard method.

Keywords: Appearance based recognition; Principle component analysis; Pose estimation

1. Introduction

Principle component analysis (PCA) is now a standard tool in computer vision. It was first introduced by Sirovich and Kirby [9], and popularized by Turk and Pentland [10] for face recognition. Murase and Nayar (see [7]) used the so-called parametric eigenspace method PCA for object recognition and pose estimation. Recent extensions of PCA include the processing of unsegmented images [8], cluttered background (see for example [2,5]) or to overcome inherent difficulties like occlusion handling and noise (see [1,6]).

Common to all these methods is the restriction on maximal two degrees of freedom (DoF) in the object pose as the search for the minimal distance in the eigenspace is not practicable for higher dimensions and the training set becomes very large. Usually, these two DoF represent the object captured from certain points along the view sphere. All the necessary information about the object is gathered. In contrast to that neither an in-plane rotation nor a horizontal or vertical translation in the image adds any new information about the object. Therefore, we call these DoF ‘non-information bearing’ as opposed to information bearing DoF. To recognize objects under non-information bearing transformations usually a search is conducted [8]. Our work addresses the inclusion and recognition of such an additional, non-information bearing third degree of freedom, namely the in-plane rotation in a set of images consisting of objects that are rotated out of the image plane. These have to be trained to be known, and we call these images views. In the proposed method, we do not need to obtain the eigenvectors by performing PCA during the training stage nor to perform the search for the third DoF in the eigenspace explicitly. We show that the additional DoF can be determined by a direct calculation.

In the following, we present the original PCA approach and introduce some notations. The image matrix \( \mathbf{X} \) is built from \( S = P \times R \) columns—the training images—and \( L \) rows

\[
\mathbf{X} = \begin{bmatrix} x_{11}, \ldots, x_{P1}, \ldots, x_{1R}, \ldots, x_{PR} \end{bmatrix}.
\]

The images \( x \in \mathbb{R}^N \) are supposed to be brightness normalized and have number of pixel \( N \). The average image \( \mathbf{c} = \frac{1}{S} \sum_{r=1}^{R} \sum_{p=1}^{P} x_{pr} \) is calculated and subtracted from the image matrix

\[
\mathbf{X} = \begin{bmatrix} x_{11} - \mathbf{c}, \ldots, x_{P1} - \mathbf{c}, \ldots, x_{1R} - \mathbf{c}, \ldots, x_{PR} - \mathbf{c} \end{bmatrix}.
\]

This so-called centered image matrix multiplied with its own transpose forms the covariance matrix

\[
\mathbf{J} = \mathbf{X} \mathbf{X}^T.
\]

Note that this matrix is symmetric, positive definite and their eigenvalues and eigenvectors are real.
Now the well-studied eigenvalue problem is

\[ \mathbf{J} \varepsilon_i = \lambda_i \varepsilon_i, \]

where \( \varepsilon_i \) is the \( i \)th eigenvector and \( \lambda_i \) its corresponding eigenvalue has to be solved.

Ueno and Kanade [11] have shown that in the case of an in-plane rotation of one image the eigenvectors of the PCA can be directly calculated by a discrete cosine transformation of the correlation. Jogan and Leonardis [3] extended this to a set of panoramic images (one non-information bearing dimension, represented in polar coordinates) and used a Fourier transform to calculate the PCA-basis. Based on this work, our approach generalizes the calculation of the eigenvectors for object recognition. We prove that the coefficients of the learned images in the eigenspace depend on the in-plane rotation angle and this dependency can be represented by an analytic function. This fact can be used for a direct calculation of the in-plane angle after the out-of-plane view has been determined. During training, our method uses an alternative approach that is much faster (and less memory demanding) than the original one. During recognition, our method has the same memory and complexity requirements as the original PCA without the in-plane rotation DoF.

In Section 2, we give the theoretical basis for the new method. We first derive the case of a single in-plane rotated view in Section 2.1 and then extend to multiple views in Section 2.2. Various experiments in Section 3 demonstrate the feasibility of the approach. Finally, we present some conclusions and outline further generalizations of the method.

2. PCA on rotated images

In the case that the rotated images are identical it is no longer necessary to separate the rotations. From the second and third equation of Theorem (1)

\[ c_{ij}^{xy} = c_{-i-j}^{xy}, \]

it follows that \( c_{ij}^{xy} \) is even. We simplify the notation \( c_{ij}^{xy} \) to \( c_{i,j} \) and \( c_{0,j} \) to \( c_i \).

2.1. One out-of-plane view

Using the property (1), Ueno and Kanada [11] showed that for one view the calculation of the eigenvectors in this case can be performed by a discrete cosine transform instead of PCA. The eigenvectors and eigenvalues of the covariance matrix are given by

\[
\mathbf{u}_k = \lambda_k^{-1/2} [x_0 x_1 \ldots x_{p-1}] \begin{bmatrix} \hat{u}_{k1} \\ \hat{u}_{k2} \\ \vdots \\ \hat{u}_{k3} \end{bmatrix} = \frac{1}{\sqrt{\lambda_k}} \sum_{h=0}^{p-1} x_h \bar{u}_{k} \]

\[
\lambda_k = \sum_{h=0}^{p-1} \cos \left( \frac{2 \pi h k}{p} \right) x_h
\]

Now we want to consider the coefficient \( f_{ij}^{xy} \) of a rotated view \( j \) represents the in-plane angle after the projection in the eigenspace (dimension \( k \)). Using the properties of (1), \( f_{ij}^{xy} \) can be expressed as

\[
f_{ij}^{xy} = \mathbf{u}_k^T x_j = \frac{1}{\sqrt{\lambda_k}} \sum_{h=0}^{p-1} \cos \left( \frac{2 \pi h k}{p} \right) x_h^T x_j
\]

\[
= \frac{1}{\sqrt{\lambda_k}} \sum_{h=0}^{p-1} \cos \left( \frac{2 \pi h k}{p} \right) c_{h-j}.
\]

Let us define the following function, which allows a transition to infinitesimal angular differences

\[
F_k^x(\phi) := \frac{f_{ij}^{xy}(\phi)}{P} = \frac{1}{2 \sqrt{\lambda_k}} \sum_{h=0}^{p-1} \cos(k \phi_h) c(\phi_h - \phi_h) \frac{2 \pi}{P}.
\]

Theorem 2. If the number of in-plane rotated images is infinite, respective the angular distance \( \Delta \phi \) distance infinitesimal, the following holds

\[
F_k^x(\phi) = \frac{1}{2 \sqrt{\lambda_k}} a_k \cos k \phi.
\]

(Proof in Appendix B).

This result allows to calculate the parametric representation of the learned images in the eigenspace and to
determine the amplitude of the resulting trigonometric function. In practical applications we need to consider the finite value of \( P \). It follows from
\[
\lim_{P \to \infty} \frac{f^k_P(\varphi)}{\frac{P}{2\sqrt{k}} a_k \cos k\varphi} = 1, \quad \cos k\varphi \neq 0
\]
that the function \( f^k_P \) is asymptotically similar to
\[
f^k_P \sim \frac{P}{2\sqrt{k}} a_k \cos k\varphi,
\]
if \( \Delta \varphi \) is small, respectively, \( P \) high. We demonstrate in the experimental section that this condition holds already for small values of \( P \).

It is interesting that our proof is also valid in the discrete case when the correlation function is known only at \( P \) sampling points. In this case we have to interpolate these points by discrete Fourier series, respective their cosine parts. The new interpolation function \( c_P(\varphi) \) is defined by
\[
c_P(\varphi) = \sum_{\nu=0}^{P-1} a_{\nu,P} \cos \nu\varphi,
\]
and the Fourier coefficients by
\[
a_{\nu,P}(\varphi) = \frac{2}{P} \sum_{\mu=0}^{P-1} c(\varphi_\mu) \cos \nu\varphi_\mu.
\]

In the infinitesimal case both expressions equal those of the continues correlation function
\[
c(\varphi) = \lim_{P \to \infty} c_P(\varphi) = \sum_{\nu=0}^{\infty} a_{\nu} \cos \nu\varphi, \quad \text{with}
\]
\[
a_{\nu} = \lim_{P \to \infty} a_{\nu,P} = \frac{1}{\pi} \int_{0}^{2\pi} c(\varphi) \cos \nu\varphi \, d\varphi.
\]

However, this is not sufficient for the proof because the sum in Eq. (B.1) is no more Riemann. But nevertheless because, \( c \) is equal to \( c_P \) at the \( P \) sampling points, the proof is also valid for the discrete case.

One can show easily that always two eigenvectors with the same eigenvalue exist (\( \delta \)th and \((P-\delta) \)th). To provide an accurate pose estimation, it is useful to introduce a phase-shift for the second eigenvector
\[
\mathbf{u}_{\delta-\delta-k} = \frac{1}{\sqrt{k}} \sum_{h=0}^{P-1} \cos \left( \frac{2\pi h(k-\delta)}{P} \right) x_h
\]
\[
= \frac{1}{\sqrt{k}} \sum_{h=0}^{P-1} \sin \left( \frac{2\pi h\delta}{P} \right) x_h.
\]

The continuous parametric representation for the shifted dimensions can be expressed by
\[
\mathbf{u}_k^\delta = \frac{1}{2\sqrt{k}} a_k \sin k\varphi
\]
because of
\[
\int_{0}^{2\pi} \sin k\varphi \sin \nu\varphi \, d\varphi = \begin{cases} 
0, & \text{for } k \neq \nu, \\
\pi, & \text{for } k = \nu \neq 0,
\end{cases}
\]
\[
\int_{0}^{2\pi} \cos k\varphi \sin \nu\varphi \, d\varphi = 0 \forall k, \quad \nu \in \mathbb{N}_0.
\]

2.2. In-plane rotated objects in several out-of-plane views

Now we consider the case of several images (e.g. different out-of-plane rotations) that are in-plane rotated. The efficient construction of a basis was first described by Jogan and Leonardis [3].

The training set consists of \( R \) images each of them is rotated \( P \)-times about an axis perpendicular to the image plane (in-plane). It does not matter whether the training set consists of several objects or just several views of the same object. \( x_i \) denotes the \( i \)th image in-plane rotated about the angle \( \theta = 2\pi h/P \). The image matrix is
\[
X = [x_0, x_1, \ldots, x_{P-1}, x_0, \ldots, x_{P-1}, \ldots, x_0, \ldots, x_{P-1}].
\]

The inner product matrix is
\[
\mathbf{J} = X^TX = \begin{bmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1R} \\
Q_{21} & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
Q_{R1} & \cdots & \cdots & Q_{RR}
\end{bmatrix},
\]
and can be represented by a symmetric matrix consisting of several circulant blocks \( Q_{lm} \), see Theorem 1.

We want to solve the eigenvalue problem
\[
\mathbf{J}\vec{\nu} = \mu\vec{\nu}.
\]

As shown in [3] a common system of eigenvectors exist for circulant matrices, namely the Fourier series
\[
\mathbf{u}_k = \begin{bmatrix} 1 \\
e^{-2\pi i(k/P)} \\
\vdots \\
e^{-2\pi i((P-1)/P)}
\end{bmatrix}, \quad \text{for } k = 0, 1, \ldots, P-1.
\]

The eigenvectors of the whole matrix can be represented by
\[
\mathbf{v}_k = \begin{bmatrix} \alpha_k^1 \mathbf{u}_k \\
\vdots \\
\alpha_k^P \mathbf{u}_k
\end{bmatrix}, \quad \text{for } k = 0, \ldots, P-1.
\]
This allows to pose the new eigenvalue problem

\[ \mathbf{A}^k \mathbf{\alpha}^k = \mu^k \mathbf{\alpha}^k \]

with

\[ \mathbf{A}^k = \begin{bmatrix} \lambda_{11}^k & \lambda_{12}^k & \cdots & \lambda_{1R}^k \\ \lambda_{21}^k & \lambda_{22}^k & \cdots & \lambda_{2R}^k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{R1}^k & \lambda_{R2}^k & \cdots & \lambda_{RR}^k \end{bmatrix} , \]

and

\[ \mathbf{\alpha}^k = \begin{bmatrix} \alpha_1^k \\ \alpha_2^k \\ \vdots \\ \alpha_R^k \end{bmatrix} . \]

The eigenvectors can be represented by

\[ \mathbf{v}_{kl} = \frac{1}{\sqrt{\mu_{kl}}} \sum_{\nu=1}^{R} \mathbf{\alpha}_{\nu}^k \sum_{j=0}^{P-1} e^{-2\pi i (\nu p)/P} \mathbf{x}_i^j . \]

Our goal is to represent the coefficient \( f_{kl}^p(j) \) of the training view \( 1 < p < R \) in the \( k \)th dimension of the eigenspace (meaning we project it on the \( k \)th eigenvector, this is complex so its transpose has to be complex conjugated) depending on the in-plane rotation \( 0 < j < P-1 \)

\[ f_{kl}^p(j) = v_{kl}^j (\mathbf{X}) = \frac{1}{\sqrt{\mu_{kl}}} \sum_{\nu=1}^{R} \mathbf{\alpha}_{\nu}^k \sum_{h=0}^{P-1} e^{-2\pi i (\nu p)/P} e^{i2\pi j \mathbf{x}_h} , \]

\[ = \frac{1}{\sqrt{\mu_{kl}}} \sum_{\nu=1}^{R} \mathbf{\alpha}_{\nu}^k \sum_{h=0}^{P-1} e^{-2\pi i (\nu p)/P} e^{i2\pi j \mathbf{x}_h} , \]

for the last identity the last property of Eq. (1) was used.

Similar to Section 2.1 one can prove the theorem.

**Theorem 3.** If the number of in-plane rotated images is infinite, respectively, the angular distance \( \Delta \phi \) distance infinitesimal, the following holds

\[ F_{kl}^p(\phi) = \lim_{P \to \infty} F_{kl}^p(\phi) = \lim_{\Delta \phi \to 0} \frac{f_{kl}^p(\phi)}{\sqrt{P}} \]

\[ = \frac{1}{\sqrt{\mu_{kl}}} \sum_{\nu=1}^{R} \mathbf{\alpha}_{\nu}^k e^{i2\pi \phi} s_{kl}^\phi , \]

with

\[ s_{kl}^\phi = \frac{P}{\sqrt{\mu_{kl}}} \sum_{\nu=1}^{R} \mathbf{\alpha}_{\nu}^k e^{i2\pi \phi} \]

(For a proof see Appendix C).

That means in the case of infinitesimal angular differences of the in-plane rotated images the coefficients of the trained images in the eigenspace can be described by circles in the complex plane. As shown in Section 2.1 for finite \( P \), the function \( f_{kl}^p(h) \) is asymptotic similar to

\[ f_{kl}^p(h) \sim \frac{1}{\sqrt{\mu_{kl}}} \sum_{\nu=1}^{R} \mathbf{\alpha}_{\nu}^k e^{i2\pi \phi} , \]

\[ \text{with} \quad a_{\nu}^p = \frac{1}{P} \sum_{\mu=0}^{P-1} e^{i2\pi \phi} . \]

For the calculation a discrete Fourier series is used for the expansion of the correlation function

\[ c_{\nu}^p(\phi) = \sum_{\nu=0}^{P-1} a_{\nu}^p e^{i2\pi \phi} . \]

Please note that our notation assumes the description of the in-plane rotation as a circular shift of a matrix, nevertheless, is the polar angle representation just used to simplify the notation. The same construction is applicable also to images in Cartesian coordinates.\(^1\) The same proof is also immediately applicable for images that are circularly shifted in \( x \) or \( y \) direction, when the rotation matrix \( T \) in (1) is replaced by the shift operator.

3. Experiments

We conducted several experiments to show the feasibility of our method. One of the most important questions to answer is the sensitivity on the ‘infinitesimal’ constraint our proof is based on. The training set consists of one object captured in \( 10^6 \) steps of out-of-plane rotation, see Fig. 1. The eigenspace dimension was 12 for the out-of-plane and four for the in-plane rotation. Table 1 shows the pose estimation results depending on the number of in-plane rotated images used to construct the basis. This result shows that using more than 18 images for

\(^1\) As the experiments will demonstrate this also holds for interpolated images due to the rotation.
the basis construction (corresponding to an angular distance of $20^\circ$) we obtain a very high accuracy on the pose estimation. Therefore, the supposition of infinitesimal distances in our proof is justified.

To make this even more visible, Fig. 2 compares the theoretical representation to the numerically calculated one.

We used only 10 in-plane rotated images so that the slight deviations are visible.

The property that always two complementary eigenvectors exist for an eigenvalue results from the derivation of the formulas. The question is what the second eigenvector contributes to the pose estimation performance. The first basis (full base) includes all eigenvectors when performing the whole base building process as described in Section 2, the second base (half base) only those eigenvectors belonging to different eigenvalues. As training set we used 10 distance for out-of-plane rotation and 20 in-plane rotation. It is expected that the accuracy of the out-of-plane estimation depends only on the number of eigenvectors with different eigenvalues (we call that effective eigenspace dimensions), the in-plane angle estimation is improved by using both eigenvectors. In Fig. 3, one can see

Table 1

<table>
<thead>
<tr>
<th>Number</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>18</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{e}_{oop}$ (degrees)</td>
<td>48.075</td>
<td>43.015</td>
<td>25.600</td>
<td>14.450</td>
<td>8.385</td>
<td>2.161</td>
<td>2.172</td>
<td>2.178</td>
</tr>
<tr>
<td>$\hat{e}_{ip}$ (degrees)</td>
<td>17.348</td>
<td>7.117</td>
<td>1.557</td>
<td>1.037</td>
<td>0.785</td>
<td>0.361</td>
<td>0.372</td>
<td>0.357</td>
</tr>
</tbody>
</table>

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the mean absolute error for both bases depending on the number of eigenspace dimensions. It is clearly visible that using twice as much dimension (and complexity) in the case of the full base does not enhance the pose estimation accuracy, the optimal number of effective eigenspace dimensions is equivalent for both bases and amounts to 13.

The mean absolute errors for the in-plane angle estimation can be seen in Fig. 4. As expected, the minimal error of the full base is slightly lower then the error of the half base, but with the cost of double complexity. Additionally, using the second eigenvector does not ameliorate the result as its frequency \( k \) equals zero; therefore, all in-plane angles of one view project to the same point in the eigenspace.

There are two different ways to estimate the in-plane angle. One can calculate it directly using Eq. (5) or organize it as a search. The first possibility is much faster but two problems arise. First of all only a few dimensions have the frequency \( k = 1 \) that allows the angle determination directly. Higher frequencies are mapped multiple times on \( 360^\circ \) and it is not obvious how to combine them in a least square sense to ameliorate the angle determination. Secondly, weighting of different dimensions must be realized. Those with a larger radius should contribute more to the result. The second approach the (traditional) search is slower, but solves both problems mentioned above implicitly. Please note that only the in-plane angles have to be determined by search because different views can be treated independently.

We tried to overcome these difficulties by combining both methods. The proposed in-plane estimation algorithm estimates the angle roughly by directly calculating the angle with Eq. (5) from a dimension with frequency \( k = 1 \), and uses the additional dimensions to ameliorate the result by a traditional search in an interval about the first estimate. This speeds up the whole procedure considerably but preserves the accuracy of the in-plane angle estimation, as can be seen in Fig. 5. By using more eigenvectors then the optimal number \( k_{\text{opt}} \) (in our experiment \( k_{\text{opt}} = 4 \)) the accuracy decreases fast because of the diminishing size of the radii.

For all the experiments hitherto we have used images in polar coordinates (see Fig. 6) to meet the assumptions of the proof. Using images in Cartesian coordinates leads to a resulting in-product matrix, Eq. (3), that is not completely circulant anymore but shows numerical interferences due to the necessary interpolations. To evaluate this difference we made another experiment using Cartesian images (basis of \( 10^\circ \) out-of-plane angle, \( 20^\circ \) in-plane angle, cubic interpolation). The results of the out-of-plane and in-plane estimation depend on the number of dimensions in the eigenspace and can be seen in Figs. 7 and 8. The estimation is in practice not degraded by using Cartesian images. Because (from a mathematical point of view) the image set has completely changed, the new optimum for the out-of-plane estimation lies at 11 dimensions and three for the in-plane angle.

Another assumption we implicitly made is that we know the rotation center. Segmentation problems could...
hinder a repeatable center estimation. The influence of the rotation center on the pose estimation is shown in Figs. 9 and 10. The center was displaced by the denoted number of pixels \( n_p \) to the right and up, so the actual shift is \( \sqrt{2} n_p \). As it was to expect the method is quite sensitive to shifts of the center and is not applicable anymore if the shift exceeds two pixels. But it should be noted that this is not a limitation of our estimation method, but of the eigenspace representation in general. For a possible solution of this shortcoming we plan to include also additional degrees of freedom like the \( x \)- and \( y \)-translation.

In the last experiment, we show that our method is also able to cope with a larger image set consisting of nine objects each taken with 10\(^\circ\) out-of-plane rotation and 10\(^\circ\) in-plane rotation—in summary 5832 images. All images are represented in Cartesian coordinates. The result is a mean absolute out-of-plane error of 2.21\(^\circ\) and a mean absolute in-plane error of 2.052\(^\circ\). The number of false classified objects can be seen in Fig. 11. These results are achieved with 11 out-of-plane and three in-plane eigendimensions.

4. Conclusion

We propose an extension of the original PCA to include the capability to cope with one additional degree of freedom provided that it does not convey additional information about the object—as it is the case for in-plane rotation. A similar reasoning holds for scaling, if the change of the distance to the object is small compared to the absolute distance. An analytic form for the eigenspace coefficients is derived that allows the determination of the additional DoF independently from the conventional out-of-plane pose and directly without explicitly performing the usual search. In various experiments we demonstrated that the method is feasible. The method can also be applied to horizontal and vertical shifts. In our future work we plan to extend it to multiple DoF and also to other linear transformation like linear discriminant analysis (LDA).

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We thank R. Heersink, Institute for Mathematic, Graz University of Technology for his kind help regarding the mathematical formulation of the theorems.

Appendix A. Proof of Theorem 1

To prove the proposed theorem we need to specify the rotation matrix \( T \). Besides the obvious fact that it has to shift
lines, it must also fulfill the following properties
\[ T^{+j} = T^T \quad T^T = T^{-i} \quad T^{+M} = T^i. \] (A.1)

An example for such a matrix is
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\] (A.2)

One can see easily by substitution that the property (A.1) is fulfilled. The actually used matrix must have the same number of columns as there are rows in the image.

Now we start with the first property of Theorem 1, it follows from Eq. (A.1) and
\[
c_{ij}^{xy} = (T_{x0}^{i})^T T_{x0}^{j} y_0 = (T_{x0}^{i})^T (T_{x0}^{p-j}) (T_{x0}^{p}) y_0 = c_{p-j}^{x}.
\]

The second property follows directly from the definition of the correlation. To prove the third property again (A.1) is used
\[
c_{ij}^{xy} = (T_{x0}^{i})^T T_{x0}^{j} y_0 = x_0^T (T_{x0}^{i})^T (T_{x0}^{j}) y_0 = x_0^T (T_{x0}^{j}) y_0 = c_{j}^{x}.
\]

Appendix B. Proof of Theorem 2

To prove Theorem 2 we start from Eq. (2). After a variable transformation the \( j \)- and \( h \)th picture can be identified by its (discrete) in-plane angle \( \phi = j(2\pi/P) \) and \( \phi_h = h(2\pi/P) \) and we can write (2) by
\[
f_k(\phi_r) = \frac{1}{\sqrt{\lambda_k}} \sum_{h=0}^{P-1} \cos(k \phi_h) c(\phi_h - \phi_r).
\]

In the following, we prove that in the case of infinitesimal angular distance \( \Delta \phi \) the resulting function \( f_k \) for the image representation must also be of the same period as the chosen base function for the eigenvectors. For this proof we interpret the eigenspace representation as function series depending on the number of used in-plane rotated images \( P \)
\[
f_k(\phi_r) = \frac{1}{\sqrt{\lambda_k}} \sum_{h=0}^{P-1} \cos(k \phi_h) c(\phi_h - \phi_r)
\]
\[
= \frac{1}{\sqrt{\lambda_k}} \frac{P-1}{P} \sum_{h=0}^{P-1} \cos(k \phi_h) c(\phi_h - \phi_r) \frac{2\pi}{P}.
\]

Now a new function is defined
\[
F_k(\phi_r) := \frac{f_k(\phi_r)}{P} \frac{1}{2\pi\sqrt{\lambda_k}} \sum_{h=0}^{P-1} \cos(k \phi_h) c(\phi_h - \phi_r) \frac{2\pi}{2\phi}.
\] (B.1)

We represent the correlation function by Fourier series
\[
c(\phi) = \sum_{r=0}^{\infty} a_r \cos r \phi
\]
with Fourier coefficients
\[
a_0 = \frac{1}{2\pi} \int_0^{2\pi} c(\phi) d\phi, \quad a_r = \frac{1}{2\pi} \int_0^{2\pi} c(\phi) \cos r \phi d\phi, \quad r = 1, 2, 3, \ldots
\]

It follows that
\[
c(\phi - \phi) = \sum_{r=0}^{\infty} a_r \cos r(\phi - \phi) + \sin r \phi \sin \phi
\] (B.2)

Now the continuous formulation can be rewritten (using Eq. (B.2)) as
\[
F_k(\phi) = \lim_{P \to \infty} F_k^\phi(\phi)
\]
\[
(\Delta \phi \to 0)
\]
\[
= \frac{1}{2\pi\sqrt{\lambda_k}} \int_0^{2\pi} \cos(k \phi) c(\phi - \phi) d\phi
\]
\[
= \frac{1}{2\pi\sqrt{\lambda_k}} \sum_{r=0}^{\infty} a_r \cos(k \phi) \cos r \phi \cos r \phi + \sin r \phi \sin r \phi) d\phi
\]
\[
= \frac{1}{2\pi\sqrt{\lambda_k}} \sum_{r=0}^{\infty} a_r \cos(k \phi) \cos r \phi \cos r \phi + \sin r \phi \sin r \phi) d\phi,
\] (B.3)

for the reordering of the sum and the integral we have to assume uniform convergence.

All integrals are zero with the exception of \( \nu = k \)
\[
\int_0^{2\pi} \cos k \phi \cos r \phi d\phi = 0 \quad \text{for} \ k \neq \nu,
\]
\[
\pi, \quad \text{for} \ k = \nu \neq 0,
\]
so the result is
\[
F_k(\phi) = \frac{1}{2\sqrt{\lambda_k}} a_k \cos k \phi.
\]
Appendix C. Proof of Theorem 3

We start from Eq. (4). The variables $i, j$ are substituted by $\phi_j = j(2\pi/P)$ and $\phi_h = h(2\pi/P)$, so the new discrete representation is

$$f_{kl}^p(\phi_j) = \frac{1}{\sqrt{P\mu_{kl}}} \sum_{r=1}^{R} \sum_{h=0}^{P-1} \alpha_{kl} e^{ik\phi_h} c_{r,\phi}(\phi_j - \phi_h).$$

We define a new function and assume that the angular difference between the in-plane rotated images infinitesimal ($P \to \infty$, i.e. $\Delta \phi \to 0$) so the lines are

$$F_{kl}^p(\phi_j) = \lim_{P \to \infty} \frac{f_{kl}^p(\phi_j)}{\sqrt{P}} = \lim_{P \to \infty} \frac{1}{2\pi \sqrt{\mu_{kl}}} \sum_{r=1}^{R} \sum_{h=0}^{P-1} \alpha_{kl} e^{ik\phi_h} c_{r,\phi}(\phi_j - \phi_h) \frac{2\pi}{P} \Delta \phi$$

$$= \frac{1}{2\pi \sqrt{\mu_{kl}}} \sum_{r=1}^{R} \sum_{h=0}^{P-1} \alpha_{kl} e^{ik\phi_h} c_{r,\phi}(\phi_j - \phi_h) \int_0^{2\pi} c_{r,\phi}(\phi) d\phi.$$

For the correlation function it follows from Eq. (1) that it is periodic and can be represented by complex Fourier series

$$c_{r,\phi}(\phi) = \sum_{r=-\infty}^{\infty} d_{r,\phi} e^{ir\phi},$$

and the Fourier coefficients

$$d_{r,\phi} = \frac{1}{2\pi} \int_0^{2\pi} c_{r,\phi}(\phi) e^{-ir\phi} d\phi.$$

The structure of the proof is comparable to that of Theorem 2. First we evolve $c_{r,\phi}(\phi - \phi)$ in Fourier series

$$c_{r,\phi}(\phi - \phi) = \sum_{r=-\infty}^{\infty} d_{r,\phi} e^{ir(\phi - \phi)}.$$

Now we can write Eq. (C.1) as

$$F_{kl}^p(\phi_j) = \frac{1}{2\pi \sqrt{\mu_{kl}}} \sum_{r=1}^{R} \alpha_{kl} \int_0^{2\pi} e^{ik\phi} \sum_{r=-\infty}^{\infty} d_{r,\phi} e^{ir(\phi - \phi)} d\phi.$$

in the case of uniform convergence that is assumed to hold the order of the integration and the sum can be changed

$$F_{kl}^p(\phi_j) = \frac{1}{2\pi \sqrt{\mu_{kl}}} \sum_{r=1}^{R} \alpha_{kl} \sum_{r=-\infty}^{\infty} d_{r,\phi} \int_0^{2\pi} e^{ir(\phi - \phi)} d\phi.$$

For the integral holds [4]

$$\int_0^{2\pi} e^{i(\phi - \phi)p} d\phi = \begin{cases} 0, & \text{for } k - v \neq 0, \\ 2\pi, & \text{for } k - v = 0, \end{cases}$$

only the term $v=k$ of the sum is not zero. Therefore

$$F_{kl}^p(\phi) = \frac{1}{\sqrt{\mu_{kl}}} \sum_{r=1}^{R} \alpha_{kl} d_{r,\phi} e^{ir\phi} = s_{kl} e^{ir\phi},$$

with

$$s_{kl} = \frac{1}{\sqrt{\mu_{kl}}} \sum_{r=1}^{R} \alpha_{kl} d_{r,\phi} \tan \left( \frac{1}{2} \sum_{r=1}^{R} \alpha_{kl} d_{r,\phi} \right).$$

References


